The quantum measurement process in an exactly solvable model

Armen E. Allahverdyan[1,2)], Roger Balian[3)] and Theo M. Nieuwenhuizen[1)*]
[1)] *Institute for Theoretical Physics, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*
[2)] *Yerevan Physics Institute, Alikhanian Brothers St. 2, Yerevan 375036, Armenia and*
[3)] *SPhT, CEA-Saclay, 91191 Gif-sur-Yvette cedex, France*

An exactly solvable model for a quantum measurement is discussed which is governed by hamiltonian quantum dynamics. The z-component $\hat{s}_z$ of a spin -$\frac{1}{2}$ is measured with an apparatus, which itself consists of magnet coupled to a bath. The initial state of the magnet is a metastable paramagnet, while the bath starts in a thermal, gibbsian state. Conditions are such that the act of measurement drives the magnet in the up or down ferromagnetic state according to the sign of s_z of the tested spin. The quantum measurement goes in two steps. On a timescale $1/\sqrt{N}$ the off-diagonal elements of the spin's density matrix vanish due to a unitary evolution of the tested spin and the N apparatus spins; on a larger but still short timescale this is made definite by the bath. Then the system is in a 'classical' state, having a diagonal density matrix. The registration of that state is a quantum process which can already be understood from classical statistical mechanics. The von Neumann collapse and the Born rule are derived rather than postulated.

It is astonishing that after one century of success of the quantum description of nature, its foundations are as mysterious as ever [1],[2],[3],[4]. To determine the precise meaning of a wavefunction (or, more generally, a density matrix), a fundamental understanding of the quantum measurement process is required, since this is the only point of contact between theory and experiment.

To investigate the matter, several models have been proposed[5], which did not converge to a unique picture. Here we discuss an exactly solvable model[6], which retains all properties of realistic measurements and from which the general structure, found before in a more complicated bosonic model[7], can be read off.

As foreseen on general grounds[8],[9], the measurement appears to take place in two steps: on a quantum timescale $\tau_{\rm c} \ll \hbar/T$, disappearence of off-diagonal elements of the spin's density matrix occurs (vanishing of Schrödinger cat terms), while on a timescale $\tau_{\rm reg} \gg \hbar/T$ the registration of the measurement occurs. The registration is analogous to the measurement of an ensemble of 'classical' Ising spins s_z taking the values $+1$ or -1.

As we shall discuss in the conclusion, our solution for the measurement problem is compatible with the statistical interpretation of quantum mechanics, as a theory that describes ensembles. It rules out several competing interpretations.

Classical measurement

The analysis of the quantum measurement, to be discussed below, appears to exhibit some classical features. For this reason, and also for its own sake, we first explain how to measure a classical Ising spin (classical two-state system), which is in a definite state $s_z = \pm 1$. It is known that some classical systems may indeed be approximately described as a two-state object, e.g. a classical brownian particle in a double-well potential with well-separated minima and a steep potential barrier in between.

Our apparatus (A) consists of a magnet (M) coupled to a phonon bath (B). The magnet contains N Ising spins $\sigma_z^{(n)} = \pm 1$ having a mean-field interaction between all quartets

$$H_{\rm M} = -\frac{J}{4N^3}\sum_{ijkl=1}^{N} \sigma_z^{(i)}\sigma_z^{(j)}\sigma_z^{(k)}\sigma_z^{(l)} = -\frac{1}{4}NJ\underline{m}^4, \quad (1)$$

with $\underline{m} = (1/N)\sum_n \sigma_z^{(n)}$ denoting the fluctuating magnetization. In the standard Curie-Weiss model all pairs would be coupled, and a second order phase transition occurs. The quartic interaction has been chosen in order to have a first order transition, as it happens in a bubble chamber, where an oversaturated liquid creates droplets of its stable phase, the gas, when triggered by a particle. In general, the apparatus should amplify the microscopic signal and go to a stable pointer state so as to allow reading at an arbitrary moment. These conditions can indeed be met when it starts in a metastable state.

The interaction between the tested system S and the apparatus

$$H_{\rm SA} = -g s_z \sum_n \sigma_z^{(n)} = -g s_z N\underline{m}, \quad (2)$$

is turned on at $t = 0$, the beginning of the measurement, and turned off at, say, $t_{\rm f}/2$, after which the apparatus is left to relax untill the final time $t_{\rm f}$.

Initially the magnet starts in the paramagnetic state: each spin has chance $\frac{1}{2}$ to be up or down, implying a vanishing average magnetization, $m \equiv \langle \underline{m} \rangle = 0$.

At a critical temperature $T_{\rm c}$ the magnet undergoes a phase transition to one among two states with magnetization $m_\uparrow > 0$ and $m_\downarrow = -m_\uparrow < 0$. Due to the quartic interactions (2), it is a first order phase transition. The free

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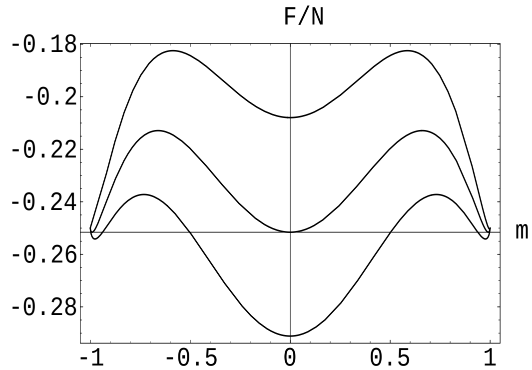

FIG. 1: Free energy of the magnet as function of m. Lower curve: at large $T = 0.42\,J$ the paramagnet $m = 0$ has lowest free energy. Middle curve: at $T_c = 0.362949\,J$ the local minima $m = 0$, $m_\uparrow = 0.990611$ and $m_\downarrow = -m_\uparrow$ become degenerate. Upper curve: Below T_c, here $T = 0.3\,J$, the paramagnet is metastable, while the minima $m_\uparrow$, $m_\downarrow$ are stable. In the measurement the magnet starts in the metastable state and ends up in one of the stable states.

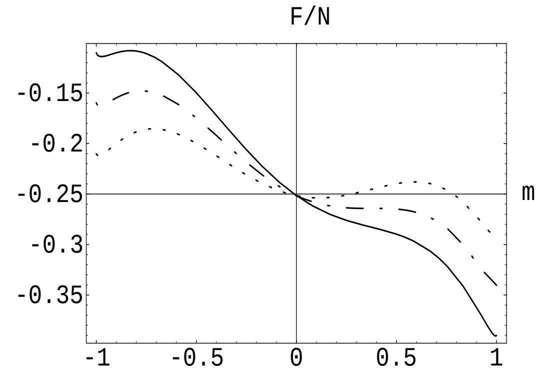

FIG. 2: A free energy barrier can be overcome by the coupling. Here $T = T_c$ and $s_z = +1$. Dotted curve: the small coupling $g = 0.04\,J$ does not suppress the barriers. The setup cannot bring the magnetization from $m = 0$ to the minimum near $m = 1$. Dash-dotted curve: at the critical value $g_c = 0.09035\,J$ the barrier near $m = 0.5$ is just suppressed. Full curve: at large coupling, $g = 0.12\,J$, there is no barrier and m will end up in the minimum to register the measurement. For $s_z = -1$ the left barrier would be suppressed.

energy, $F = U - TS$, is simply derived, owing to the fact that in this model the mean field approximation becomes exact for large N. The energy being obvious, one needs the entropy $S = \log \Omega$. Since the degeneracy of states with magnetization m equals $\Omega = N!/[(N_+)!\ (N_-)!]$, where $N_\pm = \frac{1}{2}(1 \pm m)N$, one gets immediately

$$\frac{F}{N} = -\frac{Jm^4}{4} - gs_z m - T\left(\frac{1+m}{2}\ln\frac{2}{1+m} + \frac{1-m}{2}\ln\frac{2}{1-m}\right), \tag{3}$$

At $g = 0$ and for T below $T_c = 0.362949\,J$, the paramagnet $m = 0$ is still metastable, see Fig. 1. It is here that the setup lends itself as an apparatus for a measurement: by starting in the metastable paramagnet this constitutes the magnetic analogue of the metastable oversaturated liquid of a bubble chamber.

At time $t = 0^+$ the coupling g between the tested spin and the apparatus is turned on, which puts the magnet in an external field $gs_z = \pm g$, see Eq. (2). If g is large enough and $s_z = +1$, the interaction suppresses the barrier near $m = 0.6$, see Figure 2, while for $s_z = -1$ it will suppress the one near $m = -0.6$. This is the magnetic analogue of a bubble in a bubble chamber, where a supercritical liquid is triggered to bubbles of its gas state by a tested particle.

Let us denote by $\uparrow$ and $\downarrow$ the $s_z = \pm 1$ cases. With the field turned on, the magnetization will move from $m = 0$ to the minimum of F. This is possible due to a weak coupling to the bath, which allows dumping of the excess energy in the bath. In a classical approach one avoids going into details of the bath by assuming a Glauber-type of dynamics for the spins of the magnet. Using this, the dynamics of m has been coined on the basis of detailed balance alone[10], but that is not enough to fix it. In a proper quantum mechanical treatment, one may consider the model where all three spin components of all N apparatus spins are weakly coupled to independent Ohmic bosonic subbaths (sets of harmonic oscillators). The proper dynamics then appears to be:

$$\dot{m} = \gamma h\left(1 - \frac{m}{\tanh \beta h}\right), \quad h = gms_z + Jm^3, \tag{4}$$

where $\gamma \ll 1/\hbar$ is a small parameter charactering the weak coupling to the bath. For $s_z = +1$, m will go to the right, see Fig. (2). When m has approached the minimum $+m_* \approx 0.994$, it remains stably close to 1 whether S and A are coupled or not. After decoupling the apparatus ($g \to 0$), m moves slightly from m^* to the $g = 0$ - minimum $m_\uparrow \approx 0.9906$. It will stay there up to a hopping time $\sim \exp(N)$; for large N this means “for ever”. Whether or not the apparatus is read off at any time (“observation”) is obviously of no significance for the measurement.

The measurement has now been performed: if s_z was $+1$, the apparatus has ended up with magnetization per spin $m_\uparrow \approx 1$, and for $s_z = -1$ the magnetization per spin went to $m_\downarrow = -m_\uparrow$, so the sign of the tested spin is amplified in the macroscopic magnetization $Nm_\uparrow$ or $Nm_\downarrow$.

If there is an ensemble of spins, repeating the measurement will allow determination of the fraction $p_\uparrow = \mathcal{N}_\uparrow/(\mathcal{N}_\uparrow + \mathcal{N}_\downarrow)$ of up-spins and the fraction $p_\uparrow = \mathcal{N}_\downarrow/(\mathcal{N}_\uparrow + \mathcal{N}_\downarrow)$ of down-spins, where $\mathcal{N}_\uparrow$ and $\mathcal{N}_\downarrow$ are the number of measurements with magnetization up and down, respectively.

Quantum measurement.

The above classical setup carries over immediately to the quantum situation. First, the Ising tested spin s_z should be replaced by the 2x2 Pauli matrix $\hat{s}_z$, and the apparatus spins $\sigma_z^{(n)}$ by $\hat{\sigma}_z^{(n)}$. The magnetization operator $\hat{m} = (1/N)\sum_n \hat{\sigma}_z^{(n)}$ will enter the Hamiltonians (1) and (3). In the Hamiltonian of the bath and the interaction Hamiltonian between the magnet spins and the bath, there will occur creation and annihilation operators for the bosons, while the interaction term involves, apart from those bosons, the Pauli operators for the spins.

The tested spin may start in an unknown quantum state, that is to say, its spin-averages $\langle \hat{s}_x\rangle$, $\langle \hat{s}_y\rangle$ and $\langle \hat{s}_z\rangle$ are unknown and arbitrary. The measurement is expected to determine $\langle \hat{s}_z\rangle$, while the information about $\langle \hat{s}_{x,y}\rangle$ is expected to get lost. On the basis where $\hat{s}_z$ is diagonal the initial density matrix $\hat{r}(0)$ has the elements $r_{\uparrow\uparrow}(0) = \frac{1}{2}(1+\langle \hat{s}_z\rangle)$, $r_{\uparrow\downarrow}(0) = \frac{1}{2}(\langle \hat{s}_x\rangle - \mathrm{i}\langle \hat{s}_y\rangle) = r^*_{\downarrow\uparrow}(0)$, $r_{\downarrow\downarrow}(0) = \frac{1}{2}(1-\langle \hat{s}_z\rangle)$.

The density matrix of the total system $\hat{\mathcal{D}}$ is initially chosen as a tensor product, $\hat{\mathcal{D}}(0) = \hat{r}(0)\otimes \hat{R}_\mathrm{A}(0)$, to express that there are initially no corrections between system and apparatus, in order to avoid any bias in the measurement. The apparatus itself also has two uncorrelated parts, magnet and bath, viz. $\hat{R}_\mathrm{A}(0) = \hat{R}_\mathrm{M}(0)\otimes\hat{R}_\mathrm{B}(0)$, where $\hat{R}_\mathrm{M}(0)$ describes the paramagnet, where each spin is independently up or down with chance $\frac{1}{2}$, i.e. $\hat{R}_\mathrm{M}(0) = 2^{-N}\Pi_n\hat{\sigma}_0^{(n)}$ with the identity matrix $(\hat{\sigma}_0^{(n)})_{ij} = \delta_{ij}$, while $\hat{R}_\mathrm{B}(0)$ is the equilibrium (Gibbs) state of the bath.

Selection of the collapse basis

The dynamics is set by the von Neumann equation $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{D}} = [\hat{H},\hat{\mathcal{D}}]$, where $\hat{H}$ is the full Hamiltonian operator, including also the bath and the coupling between magnet and bath. The state of the tested spin is $\hat{r}(t) = \mathrm{tr}_{\mathrm{M,B}}\hat{\mathcal{D}}(t)$. For its evolution the quantum version of the interaction Hamiltonian (3) provides,

$$\frac{\mathrm{d}}{\mathrm{d}t}\, r_{ij} = -gN(s_i - s_j)\,\mathrm{tr}_{\mathrm{M,B}}[\,\hat{m},\hat{\mathcal{D}}_{ij}], \tag{5}$$

where $i,j = \uparrow,\downarrow$ and $s_\uparrow = +1$, $s_\downarrow = -1$ are the eigenvalues of $\hat{s}_z$ and where the four blocks $\hat{\mathcal{D}}_{ij}$ of $\hat{\mathcal{D}}$ act in the apparatus space. It follows that the diagonal elements are conserved in time: $r_{\uparrow\uparrow}(t) = r_{\uparrow\uparrow}(0)$, $r_{\downarrow\downarrow}(t) = r_{\downarrow\downarrow}(0)$. This happens because the spin has no dynamics of its own. The conservation is a sine-qua-non condition for a reliable ideal measurement. The off-diagonal elements are not conserved since for them $s_i - s_j \neq 0$, so they are endangered and they will actually vanish.

We learn from this argument that the selection of the collapse basis is a direct consequence of the forces exerted by the apparatus on the test system: The choice of the interaction Hamiltonian sets the basis on which the density matrix of the system diagonalizes. Zurek has claimed that the selection would be imposed by the coupling to the environment[11], even though the difficulties to control these couplings make it an undesired candidate for such an important issue. The above argument does in no way invoke the environment and thus rules out Zurek's picture.

Disappearance of Schrödinger cats

At $t=0$ the coupling between system and apparatus is turned on. Let us consider very early times, where both the spin-spin interactions and the spin-bath interactions are still ineffective. The problem is then simply the evolution of N independent apparatus spins, not coupled to the bath, in a field arising from the tested spin. This means that within $\hat{\mathcal{D}}_{\uparrow\downarrow}$ the density matrix of each spin evolves as $\hat{\sigma}_0^{(n)} = \mathrm{diag}\ (1,1) \to \mathrm{diag}\ (e^{2\mathrm{i}gt/\hbar},\ e^{-2\mathrm{i}gt/\hbar})$, implying

$$r_{\uparrow\downarrow}(t) \equiv \mathrm{tr}_{\mathrm{M,B}}\hat{\mathcal{D}}_{\uparrow\downarrow}(t) = r_{\uparrow\downarrow}(0)[\cos\frac{2gt}{\hbar}]^N. \tag{6}$$

For short times this produces a gaussian decay, $r_{\uparrow\downarrow}(0)\ \exp(-t^2/\tau_\mathrm{c}^2)$, with 'cat' time

$$\tau_\mathrm{c} = \frac{\hbar}{g\sqrt{2N}} \ll \frac{\hbar}{T}. \tag{7}$$

In the estimate we took $g\sim J\sim T$ and $N\gg 1$.

The matrix element (6) still presents recurrence peaks at $t_k = k\pi\hbar/2g$. However, provided N is large they will be suppressed by the bath, as it brings in a factor $\sim \exp(-\gamma\hbar N)$. A small dispersion in the g's is quite realistic, and it also brings a reduction $\exp(-k^2\pi^2\frac{\langle g^2\rangle - \langle g\rangle^2}{2\langle g\rangle^2}N)$.

Altogether, in the quantum coherent process which takes place on the shortest time scale (7), the Schrödinger cat hides itself if N is macroscopic, in agreement with von Neumann's postulate. The recurrence of the peaks is suppressed somewhat later, also on a short time scale, owing either to a small coupling of the macroscopic magnet with the bath or to a dispersion in the couplings g. In the first case the erasure of recurrences is an effect of the 'environment', but, contrary to what is often thought, see Zurek [11] for a recent review, the environment (bath) appears not to be the main cause.

Registration of the quantum measurement.

Once the off-diagonal sectors of the density matrix have decayed, there remain the diagonal ones, which evolve more slowly because of dumping energy in the bath. Now the bath, which we describe by a model simulating phonons, and the coupling to it have to be specified in detail. For simplicity each spin of the magnet

is assumed to have its own subbath. These subbaths are all identical but independent, consisting of harmonic oscillators in x, y and z-direction, which are coupled bilinearly to the components of the spins, and start out in their Gibbs state. The characteristic coupling constant with the Ohmic bath is γ, and weak coupling means that $\gamma \ll 1/\hbar$. Working out this quantum problem we observe complete analogy to the above description termed "classical measurement". In particular, the evolution of $m(t) = \mathrm{tr}\,\hat{m}\hat{\mathcal{D}}(t)$ is found to be given by Eq. (4) announced above. The characteristic timescale is much larger than the 'cat' time,

$$\tau_{\mathrm{reg}} = \frac{1}{\gamma g} \sim \frac{1}{\gamma J} \sim \frac{1}{\gamma T} \gg \frac{\hbar}{T}. \tag{8}$$

The physical reason is that an extensive amount of energy has to be transfered to the bath; this takes a time $\sim 1/\gamma$ since the characteristic coupling constant with the bath is γ. The stable points of the dynamics are the minima of the free energy discussed above.

The last stage, where after decoupling the apparatus (by setting $g = 0$) m is stabilized at either $m_\uparrow$ or $m_\downarrow$, proceeds as in the classical case.

In short, registration of the quantum measurement is the same as for the above classical measurement. This might perhaps have been anticipated from the fact that on the considered timescale the diagonalization has already taken place, so the density matrix is "classical".

Post-measurement state

After the measurement, at $t_{\mathrm{f}} \gg \tau_{\mathrm{reg}}$, the common state of the tested spin and apparatus is stationary and equal to

$$\hat{\mathcal{D}}(t_{\mathrm{f}}) = r_{\uparrow\uparrow}(0)|\uparrow\rangle\langle\uparrow| \otimes \hat{R}_{\mathrm{A}\uparrow}(t_{\mathrm{f}}) + r_{\downarrow\downarrow}(0)|\downarrow\rangle\langle\downarrow| \otimes \hat{R}_{\mathrm{A}\downarrow}(t_{\mathrm{f}}) \tag{9}$$

where $\hat{R}_{\mathrm{A}\uparrow}(t_{\mathrm{f}})$ is the product of a gibbsian state for the bath and of the state

$$\hat{R}_{\mathrm{M}\uparrow}(t_{\mathrm{f}}) = \hat{\rho}^{(1)}_{\uparrow\uparrow}(t_{\mathrm{f}}) \otimes \cdots \otimes \hat{\rho}^{(N)}_{\uparrow\uparrow}(t_{\mathrm{f}}) \tag{10}$$

for the magnet, and where where $\hat{\rho}^{(n)}_{\uparrow\uparrow}(t_{\mathrm{f}}) = \frac{1}{2}\,\mathrm{diag}(1 + m_\uparrow, 1 - m_\uparrow)$ is the Gibbs density matrix of spin n for the magnet, and where in the down sector one replaces $m_\uparrow \to m_\downarrow = -m_\uparrow$.

The off-diagonal sectors, called "Schrödinger cat components", have been eliminated by the initial evolution. Strictly speaking, the final state must be unitarily related to the initial state. However, the entropy of (9) is larger than the initial entropy. The solution of this paradox is the same as the solution of the paradox of irreversiblilty in statistical mechanics: Eq. (9) has been derived by using some approximations and relates to a coarse grained entropy, but it differs from the exact state only through non-observable terms. The latter involve correlations of many degrees of freedom of the apparatus which for large N become negligible for almost all observables except for the fine-grained entropy, which is conserved but of no relevance.

Born rule from statistics of pointer variables

From the result (9) for the final quantum state of the system and the apparatus we can derive the characteristic function and hence the joint probability distribution for the pointer variable and the z-component of the tested spin. This distribution is peaked for large N at two points: The average magnetization per spin m can take the two values $m_\uparrow$ or $m_\downarrow$ with respective probabilities $p_\uparrow = r_{\uparrow\uparrow}(0)$ or $p_\downarrow = r_{\downarrow\downarrow}(0)$, and these two occurrences are completely correlated with the final state up or state down of the tested system.

This means that, when an ensemble of measurements is performed, two outcomes are possible for each event. What is observed is the pointer variable m of the apparatus after the process. From (9) we can calculate the moments of magnetization per spin after the measurement, $\langle m^k\rangle(t_{\mathrm{f}}) \equiv \mathrm{tr}\,\hat{m}^k\hat{\mathcal{D}}(t_{\mathrm{f}}) = r_{\uparrow\uparrow}(0)m_\uparrow^k + r_{\downarrow\downarrow}(0)m_\downarrow^k$, which just confirms the dichotomic distribution of $m(t_{\mathrm{f}})$. We may indeed consider $P(m;t_{\mathrm{f}}) = \mathrm{tr}_{\mathrm{S,A}}\delta(\hat{m} - m)\hat{\mathcal{D}}(t_{\mathrm{f}})$, which exposes the dichotomicity most directly,

$$P(m;t_{\mathrm{f}}) = r_{\uparrow\uparrow}(0)\delta(m - m_\uparrow) + r_{\downarrow\downarrow}(0)\delta(m - m_\downarrow). \tag{11}$$

Quantum mechanics confronts us with this macroscopic relation; its application to nature should set its interpretation. It is from (11), which refers to observations about the directly observable quantity m, and moreover from the theory which shows that (9) encompasses full classical correlations between m and s_z, that interpretations of quantum mechanics can be confronted with experiments. But this is rather standard when we realize that (11) is a relation referring to the macroscopic world only. From experimental practice we know that what is observed is the pointer variable of the apparatus after an individual measurement, the final magnetization, which equals $Nm_\uparrow$ or $Nm_\downarrow$. Classical probability theory then says that we are dealing with an *ensemble of measurements on an ensemble of systems or system preparations*, and that the factor $r_{\uparrow\uparrow}(0)$ in (11) should be identified with the fraction $p_\uparrow = \mathcal{N}_\uparrow/(\mathcal{N}_\uparrow + \mathcal{N}_\downarrow)$ individual measurements ('events') where a positive magnetization $Nm_\uparrow$ is observed.

In each of these occurrences we can infer indirectly from quantum theory, relying on (9), that the tested system is prepared in the pure state with $s_z = +1$ (this is sometimes called collapse of the wavefunction or reduction of the wavepacket). This system-apparatus connection becomes more practical if, analogous to (11), we maintain about the apparatus only the information regarding its pointer variable. This is done by considering $\hat{R}(m;t_{\mathrm{f}}) = \mathrm{tr}_{\mathrm{A}}\,\delta(\hat{m} - m)\,\hat{\mathcal{D}}(t_{\mathrm{f}})$, which is a classical distribution function of the pointer variable and an operator

in the Hilbert space of the tested system. Eq. (9) yields

$$\hat{R}(m;t_\mathrm{f}) = r_{\uparrow\uparrow}(0)\delta(m-m_\uparrow)\,|\!\uparrow\rangle\langle\uparrow\!| + r_{\downarrow\downarrow}(0)\delta(m-m_\downarrow)|\!\downarrow\rangle\langle\downarrow\!|. \tag{12}$$

For practical applications this result of the measurement process defines a broad class of ideal measurements, whereas (9) describes the complete final state of the apparatus, that, however, is hardly ever tested.

It is thus the theoretical analysis of the interaction process between the system and the apparatus which allows us to regard this process as a "measurement" in which microscopic quantum information is deduced from macroscopic observation. In order to get this information on s_z in the form of an ordinary probability, we had to pay a tribute: the loss of the initial information about the off-diagonal elements $r_{\uparrow\downarrow}(0)$ and $r_{\downarrow\uparrow}(0)$. The emergence of classical probabilities is due to the macroscopic size of the apparatus, and it is accompanied by a destruction of genuinely quantum elements.

Born's rule, together with von Neumann's collapse, are thus derived for large N from the joint quantum evolution of the system and the apparatus. The measurement process, accompanied by a sorting of the outcomes of the pointer variable, can be used as a preparation of the system in a pure eigenstate of $\hat{s}_z$.

The statistical (ensemble) interpretation of quantum mechanics.

In agreement with the preceding analysis, it is natural to describe the quantum measurement by adopting the statistical interpretation put forward by Einstein, see e.g. [12], [9]. The most important aspects are: 1) A quantum state is described by a density matrix. 2) A single system does not have "its own" density matrix or wavefunction; 3) Each quantum state describes an ensemble of identically prepared systems; this also holds for a pure state $|\psi\rangle\langle\psi|$.

To give an example, in an ideal Stern-Gerlach experiment all particles in the upper beam together are described by the ket $|\!\uparrow\rangle$ or density matrix $|\!\uparrow\rangle\langle\uparrow\!|$.

In this philosophy, a quantum measurement must describe an ensemble of measurements on an ensemble of systems. This is indeed a natural interpretation of the post-measurement state (9). In doing a series of experiments, there are two possible outcomes, connected with the magnetization of the apparatus being up or down, which occur with probabilities $p_\uparrow = r_{\uparrow\uparrow}(0)$ and $p_\downarrow = r_{\downarrow\downarrow}(0)$, respectively. In each such event, the z-component of the tested spin is equal to $+1$ (up) or -1 (down), correspondingly. The quantum subensemble of spins having $\uparrow$ is described by the pure state density matrix $|\!\uparrow\rangle\langle\uparrow\!|$, or simply by the wavefunction $|\!\uparrow\rangle$. A similar statement holds for the down spins. This is von Neumann's collapse postulate, and it arises here as a physical consequence of quantum mechanics itself, taking into account that the apparatus is macroscopic.

Notice that the very same eq. (9) and its interpretation would arise from the thermodynamics of measurements on an ensemble of classical Ising spins ±1, thus merging classical and quantum measurements. In the classical case one would be accustomed to trace out the bath, which would replace $\hat{R}_{\mathrm{A}\uparrow}(t_\mathrm{f})$ by $\hat{R}_{\mathrm{M}\uparrow}(t_\mathrm{f})$, but we refrained from doing this, because of the confusion in the literature about its justification. We now see that also in the quantum situation no information is lost when taking this trace, because the off-diagonal terms $|\!\uparrow\rangle\langle\downarrow\!|$ and $|\!\downarrow\rangle\langle\uparrow\!|$ have become inobservably small anyhow.

Let us notice that the statistical interpretation makes sense at all times, at $t = 0$, during the measurement and after the measurement. Our mixed initial state of the apparatus describes a realistic preparation of the apparatus, as opposed to the often assumed pure initial states, that can in practice not be produced for any system with many degrees of freedom. We thus do not consider pure state setups - that might have their own interest - as realistic measurement setups.

Comparison with von Neumann-Wigner measurement theory

To compare our results with the standard description of the quantum measurement [1],[3],[4] is not easy, because the latter does not embody the same physics and it is based more on assumptions than on derivations. Since it has, in our view, not solved the measurement problem, the best we can do is to mention some analogies and differences. In the von Neumann-Wigner approach one assumes that the apparatus starts in a pure state $|a_0\rangle$. Typically also the tested system is assumed to start in a pure state, $|\psi\rangle = c_\uparrow|\!\uparrow\rangle + c_\downarrow|\!\downarrow\rangle$. It is then assumed[3] that in the so-called *premeasurement* stage $0 < t < \tau_\mathrm{c}$, the total wavefunction develops as $|\Psi\rangle = |a_0\rangle|\psi\rangle \rightarrow |\Psi_\mathrm{c}\rangle = U|\Psi\rangle = c_\uparrow|\!\uparrow\rangle|a_\uparrow\rangle + c_\downarrow|\!\downarrow\rangle|a_\downarrow\rangle$, where the $|a_{\uparrow,\downarrow}\rangle$ are states of the apparatus assumed to be in one-to-one correspondence with its final pointer states at t_f. The pure density operator $|\Psi_\mathrm{c}\rangle\langle\Psi_\mathrm{c}|$ differs from our Eq. (9), but it has same final marginal state for the tested system, $\hat{r}(\tau_\mathrm{c}) = |c_\uparrow|^2|\!\uparrow\rangle\langle\uparrow\!| + |c_\downarrow|^2|\!\downarrow\rangle\langle\downarrow\!|$. The state for the apparatus $\hat{R}_\mathrm{A}^{\mathrm{vNW}}(\tau_\mathrm{c}) = |c_\uparrow|^2|a_\uparrow\rangle\langle a_\uparrow| + |c_\downarrow|^2|a_\downarrow\rangle\langle a_\downarrow|$ would agree with the result obtained by tracing out the tested spin from eq. (9), if we would disregard the mixed nature of our states $\hat{R}_{\mathrm{A}\,\uparrow,\downarrow}(\tau_\mathrm{c})$. However, in order to interpret the process as a measurement, we need the off-diagonal terms in the overall state of S+A to dissapear. This is achieved in our model by the first stage of the evolution. Both this model and the von Neumann-Wigner theory involve unitary dynamics. However, in our treatment, the off-diagonal elements $|\!\uparrow\rangle\langle\downarrow\!|$ and $|\!\downarrow\rangle\langle\uparrow\!|$ vanish very rapidly (more precisely, they lead to very small, non-observable terms), due to the mixed nature of the initial state of the apparatus, while the bath or randomness of coupling is needed to make this erasure permanent. In the standard von Neumann Wigner approach, they survive because the initial state of the apparatus is pure; the question of

how to discard them has led to various interpretations of quantum mechanics. Here the disappearance of these terms appears simply as a statistical phenomenon, due to the averaging over the initial disordered state. Anyhow, a pure state of the apparatus is unrealistic on physical grounds.

A next claim of the von Neumann-Wigner approach is that the statistics of pointer variables is correctly described. But it is well known that their marginal state can arise from all kinds of sets of mixed non-orthogonal states of the form $\{|a_\alpha\rangle\langle a_\alpha|\}_{\alpha=1}^{k}$ with $k \geq 2$: $\hat{R}_{\rm A}^{\rm vNW}(\tau_{\rm c})=\sum_{\alpha=1}^{k}\lambda_\alpha|a_\alpha\rangle\langle a_\alpha|$, and these states have in general nothing to do with spin up or spin down states along the z-axis. In other words, it is unclear why the chosen apparatus measures the spin in the z-direction. This non-uniqueness of the measurement basis ('prescribed ensemble fallacy') is related to the fact that typically no interaction Hamiltonian between system and apparatus is specified, while in our approach it was vital for the selection of this basis.

Other interpretations of quantum mechanics.

Our results make some of the interpretations, that caused much dispute in the past[1],[2],[4] obsolete.

In the standard Copenhagen interpretation, it is stated that "the wave function is the most complete description of the system", in other words, each closed system has its own wavefunction, a fact denied in the statistical interpretation. A criticism of our approach might be that "the problem is not solved because somehow there is the (unknown) wavefunction of the total system", which cannot end up in a mixture. This argument sets out, however, that both the system and the apparatus start out in a pure state, which is unrealistic. We acknowledge, indeed, that it is impossible to prepare a macroscopic system in a pure state. Afterall, that would require a macroscopic number of post-measurement selections; what an experimentalist does is completely the opposite: he turns on the apparatus and waits until it has stabilized, after which the measurement is carried out.

The assumption of an underlying pure state for the whole system is unnecessary and would anyhow be problematic for describing the statistical nature of the apparatus in realistic setups. Moreover, it would prevent the appearance of a domain with classical features.

Interestingly, our quantum mechanical description of the registration bears certain classical features, an issue imagined long ago by Bohr.

The multi-universe picture was devised to suppress the possibility of collapse, a phenomenon which seems to contradict the unitarity of the microscopic quantum evolutions. We have seen, however, that the diagonalization is a real process of quantum statistical physics, which occurs owing to the necessarily large size of the apparatus; collapse in individual events is the physical realization of this. Moreover, for the case of finite but large N, our model describes a good measurement up to a long but finite time, while the multi-universe picture is silent about this situation, the only one met in practice.

Mind-body problems do not show up, because the act of observation is no more than gathering information about the classical final state of the apparatus, which has already registered the relevant data. Whether or not one observes the outcome has no effect on the system. Observation appears just as a means for selecting a statistical subensemble with well defined spin, owing to the system-apparatus correlations.

We have shown that the collapse is not caused by the environment[11], but by the coupling to the apparatus which selects the collapse basis.

Gravitation, sometimes put forward, plays no role.

Extensions of quantum mechanics, like spontaneous or stochastic localization and spontaneous collapse models, are not needed.

We find no support for interpretations that attribute a special role to pure states of the tested spin, e.g. the modal interpretation and bohmian or nelsonian mechanics.

Conclusion

The initial paramagnetic state of the apparatus, which cannot be fully specified, consists of many microstates, so a statistical description is called for, and we have retained it to account for the quantum measurement. This is possible within the statistical interpretation of quantum mechanics, which states that any quantum state describes an ensemble of systems. A theory of quantum measurements must therefore describe an ensemble of measurements on an ensemble of identically (fully or partially) prepared systems.

It was found that the off-diagonal terms (Schrödinger cat states) disappear quite fast after the start of the measurement. It goes in two steps: the disappearence proper occurs due to interaction of the tested system with the macroscopic apparatus, and later is made definite either by bath-induced decoherence or already by randomness in the interaction.

The registration of the measurement occurs in a "classical" state, a state that has already a diagonal density matrix. Here a naive classical approach and a detailed quantum approach yield the same outcome. The pointer variable ends up in a stable thermodynamic equilibrium state. Whether the outcome is observed or not is immaterial.

For a macroscopic apparatus disappearence of off-diagonal terms is almost instantaneous, yielding the basis for the postulate of von Neumann. The Born rule follows from the statistics of pointer values.

Our theory can be tested by mapping out the N-dependence.

An important issue, namely whether single events can be accounted for by quantum mechanics, has to be an-

swered negatively, since this would be incompatible with Eq. (11). Within the statistical interpretation, *quantum mechanics is a theory about the statistics of outcomes of experiments*, but it is unable to account for a single process. To describe single outcomes, a richer theory ("subquantum mechanics" or "hidden variable theory") might be dreamt of.

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